5.1. Sectional curvature of submanifolds. Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with sectional curvature $\bar{\sec}$. Let $p \in \bar{M}$ and $L \subset T\bar{M}_p$ an m-dimensional linear subspace.

1. Prove that there is some $r > 0$ such that for the open ball $B_r(0) \subset T\bar{M}_p$, the set $M := \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\bar{M}$.
2. Let g be the induced metric on M and let sec be the sectional curvature of M. Show that for $E \subset TM_p$, we have $\sec_p(E) = \bar{\sec}_p(E)$ and if L is a 2-dimensional subspace, then $\sec \leq \bar{\sec}$ on M.

Solution. 1. First, we know that there is some $r > 0$ such that the restriction of the exponential map to $B_r(0)$, i.e. $\exp_p|_{B_r(0)} \colon B_r(0) \to \exp_p(B_r(0))$, is a diffeomorphism. Furthermore, note that $L \cap B_r(0)$ is an m-dimensional submanifold of $B_r(0)$ and hence $M = \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\exp_p(B_r(0))$. Finally, as $\exp_p(B_r(0))$ is open in $\bar{M}$, it follows that M is a submanifold of $\bar{M}$ as well.

2. Let $u, v \in E$ be an orthonormal basis of $E \subset TM_p$. Then we have

$$\begin{aligned}
\sec_p(E) &= R_p(u, v, u, v) \\
&= \bar{R}_p(u, v, u, v) + \bar{g}_p(h_p(u, u), h_p(v, v)) - \bar{g}_p(h_p(u, v), h_p(u, v)) \\
&= \bar{\sec}_p(E) + \bar{g}_p(h_p(u, u), h_p(v, v)) - \bar{g}_p(h_p(u, v), h_p(u, v))
\end{aligned}$$

We now prove that $h_p(u, u) = h_p(v, v) = h_p(u, v) = 0$. Extend u, v to an orthonormal basis $e_1 = u, e_2 = v, e_3, \ldots, e_{\bar{m}}$ of $T\bar{M}_p$. Then this basis induces normal coordinates on $\bar{M}$. Hence, we have $\Gamma^k_{ij}(p) = 0$ and thus $(\bar{D}_{e_i} e_j)_p = 0$ for all i, j. This especially implies that $h_p(u, u) = h_p(v, v) = h_p(u, v) = 0$ as claimed.

Assume now that $L \subset T\bar{M}_p$ is 2-dimensional and let $q := \exp_p(x) \in M$ for $x \in L \cap B_r(0)$. By the above, we may assume that $x \neq 0$. Define $w := \frac{x}{|x|} \in TM_p$ and let c_w be the unique geodesic with $c(0) = p$ and $\dot{c}(0) = w$. Then we have $q = c_w(|x|)$ and $u := \dot{c}_w(|x|) \in TM_q$ with $|u| = 1$. Furthermore, we get

$$h_q(u, u) = \left(\frac{\bar{D}}{dt} \dot{c}_w \bigg|_{t=|x|} \right)^{\perp} = 0.$$

To compute the sectional curvature of $E = TM_q$, we extend u to an orthonormal basis u, v of E and get

$$\sec_q(E) = R_q(u, v, u, v) = \bar{R}_q(u, v, u, v) - |h_q(u, v)|^2 \leq \bar{\sec}_q(E)$$

as desired. $\square$

5.2. Codazzi equation. Let $M \subset \bar{M}$ be a submanifold of the Riemannian manifold $(\bar{M}, \bar{g})$. For the second fundamental form h of M, we define

$$(D_X^\perp h)(Y, W) := (\bar{D}_X(h(Y, W)))^\perp - h(D_X Y, W) - h(Y, D_X W),$$

where $W, X, Y \in \Gamma(TM)$. Show that the Codazzi equation

$$\big(\bar{R}(X, Y)W\big)^\perp = (D_X^\perp h)(Y, W) - (D_Y^\perp h)(X, W)$$

holds for all $W, X, Y \in \Gamma(TM)$.

Solution. As $\bar{D}_Z W = D_Z W + h(Z, W)$ for $W, Z \in \Gamma(TM)$, we get

$$\begin{aligned}
\bar{R}(X, Y)W = {}& \bar{D}_X \bar{D}_Y W - \bar{D}_Y \bar{D}_X W - \bar{D}_{[X,Y]} W \\
= {}& \bar{D}_X(D_Y W + h(Y, W)) - \bar{D}_Y(D_X W + h(X, W)) \\
& -(D_{[X,Y]} W + h([X, Y], W)) \\
= {}& D_X D_Y W + h(X, D_Y W) + \bar{D}_X(h(Y, W)) \\
& -D_Y D_X W - h(Y, D_X W) - \bar{D}_Y(h(X, W)) \\
& -D_{[X,Y]} W - h(D_X Y - D_Y X, W) \\
= {}& R(X, Y)W \\
& +\bar{D}_X(h(Y, W)) - h(D_X Y, W) - h(Y, D_X W) \\
& -\bar{D}_Y(h(X, W)) + h(D_Y X, W) + h(X, D_Y W).
\end{aligned}$$

Note that we used that D is torsion free, i.e. $[X, Y] = D_X Y - D_Y X$. Now, taking the perpendicular part, we conclude that the Codazzi equation

$$\big(\bar{R}(X, Y)W\big)^\perp = (D_X^\perp h)(Y, W) - (D_Y^\perp h)(X, W)$$

holds. □

5.3. Asymptotic expansion of the circumference. Let M be a manifold, $E \subset TM_p$ a linear 2-plane and $\gamma_r \subset E$ a circle with center 0 and radius $r > 0$ sufficiently small. Show that

$$L(\exp(\gamma_r)) = 2\pi \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right)$$

for $r \to 0$.

Solution. Let $v, w \in TM_p$ be an orthonormal basis of E. Then the circle can be parametrized by $\gamma_r(\varphi) = r(v\cos\varphi + w\sin\varphi)$. For some fixed $\varphi_0 \in [0, 2\pi]$, consider the Jacobi field $Y_{\varphi_0}(r)$ associated to the geodesic variation $V(\varphi_0, r) := \exp(\gamma_r(\varphi_0))$ of the geodesic $c_{\varphi_0}(r) := \exp(\gamma_r(\varphi_0))$. Then it holds

$$L(\exp(\gamma_r)) = \int_0^{2\pi} |Y_{\varphi_0}(r)|\, d\varphi.$$

We will now compute the Taylor expansion for $|Y_0(r)|$, all other cases are similar. Clearly, we have $Y_0(0) = 0$ and $Y_0'(0) = w$. From the Jacobi equation we also get

$$Y_0''(0) = -R\left(Y_0, c_0'\right) c_0'\Big|_{r=0} = 0.$$

Now taking the derivative of the Jacobi equation, we get

$$Y_0'''(0) = -\frac{D}{dr} R\left(Y_0, c_0'\right) c_0'\Big|_{r=0} = -R\left(Y_0', c_0'\right) c_0'\Big|_{r=0} = -R(w, v)v.$$

It follows that

$$|Y_0(r)| = r - \frac{R(w, v, w, v)}{6} r^3 + \mathcal{O}(r^4).$$

Therefore, we finally get

$$L(\exp(\gamma_r)) = \int_0^{2\pi} \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4)\right) d\varphi = 2\pi\left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4)\right),$$

as it was to show. □